Mathematics Higher Paper 1

2012

1. $u_{n+1} = 3u_n + 4$, $u_0 = 1$

$u_1 = 3 \times 1 + 4 = 7$

$u_2 = 3 \times 7 + 4 = 25$ (C)

2. $y = x^3 - 6x + 1$

$\frac{dy}{dx} = 3x^2 - 6$

When $x = -2$

$\frac{dy}{dx} = 3 \times (-2)^2 - 6 = 6$

$m_{tangent} = 6$ (D)

3. $x^2 - 6x + 14$

$(x-3)^2 - 9 + 14$

$(x-3)^2 + 5$

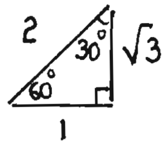
$q = 5$ (B)

4. $m = \tan\theta$

$m = \tan 150^\circ$

$m = -\tan 30^\circ$

$m = -\frac{1}{\sqrt{3}}$ (B)



5. $\cos 2a = 2\cos^2 a - 1$

$= 2 \times \left(\frac{4}{5}\right)^2 - 1$

$= 2 \times \frac{16}{25} - 1$

$= \frac{32}{25} - \frac{25}{25}$

$= \frac{7}{25}$ (A)

or.

$$\left[\begin{aligned} \cos 2a &= 1 - 2\sin^2 a \\ &= \cos^2 a - \sin a \end{aligned}\right]$$

6. $y = 3x^{-2} + 2x^{\frac{3}{2}}$, $x > 0$

$\frac{dy}{dx} = -6x^{-3} + \frac{3}{2} \times 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = -6x^{-3} + 3x^{\frac{1}{2}}$ (C)

7. $\underline{u} \cdot \underline{v} = 0$

$-3 \times 1 + 1 \times t + 2t \times (-1) = 0$

$-3 + t + (-2t) = 0$

$-t = 3$

$t = -3$ (A)

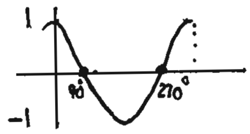
8. $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 3 \times \frac{4}{3}\pi r^2$

$\frac{dV}{dr} = 4\pi r^2$

When $r = 2$

$\frac{dV}{dr} = 4 \times \pi \times 2^2 = 16\pi$ (C)

9. 

$y = \cos x$

New curve amplitude $= 1$

translation of $\frac{\pi}{6}$ to right

translation of 1 down

$\therefore\ y = \cos\left(x - \frac{\pi}{6}\right) - 1$ (A)

10. $\vec{RP} = \vec{RS} + \vec{ST} + \vec{TP}$

$= -\underline{g} - \underline{f} + \underline{h}$ (B)

11. $\int \frac{1}{6x^2}\,dx \quad , \quad x \neq 0$

$\int \frac{1}{6} x^{-2}\,dx$

$\frac{1}{6} \quad \frac{x^{-1}}{-1} + C$

$-\frac{1}{6} x^{-1} + C$ D

12. Max value $= 5$

This occurs when

$\sin\left(x - \frac{\pi}{3}\right) = -1$

$x - \frac{\pi}{3} = \frac{3\pi}{2}$

$x = \frac{3\pi}{2} + \frac{\pi}{3}$

$x = \frac{9\pi}{6} + \frac{2\pi}{6}$

$x = \frac{11\pi}{6}$ B

13. Roots at $x = -2$ and $x = -1$

Equation of the form

$y = k(x+2)(x+1)$

$\overset{x}{(0},\overset{y}{6)}$ lies on the curve

$6 = k(0+2)(0+1)$

$2k = 6$

$k = 3$

Equation becomes

$y = 3(x+2)(x+1)$ D

14. $\int (2x-1)^{\frac{1}{2}}\,dx$

$= \frac{(2x-1)^{\frac{3}{2}}}{\frac{3}{2} \times 2} + C$

$= \frac{1}{3}(2x-1)^{\frac{3}{2}} + C$ A

15. If $\underline{u}$ is a unit vector

then $|\underline{u}| = 1$

$\underline{u} = \begin{pmatrix} 3k \\ -k \\ 0 \end{pmatrix} \quad , \quad k > 0$

$|\underline{u}| = \sqrt{(3k)^2 + (-k)^2 + 0^2} = 1$

$\sqrt{9k^2 + k^2} = 1$

$\sqrt{10k^2} = 1$

$\sqrt{10}\,k = 1$

$k = \frac{1}{\sqrt{10}}$ D

16. $y = 3\cos^4 x = 3(\cos x)^4$

$\frac{dy}{dx} = 4 \times 3(\cos x)^3 \times (-\sin x)$

$\frac{dy}{dx} = -12\cos^3 x \sin x$ C

[Chain Rule.]

17. $\underline{a} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \quad |\underline{a}| = \sqrt{3^2 + 4^2 + 0^2}$

$|\underline{a}| = \sqrt{25}$

$|\underline{a}| = 5$

$\underline{a} \cdot (\underline{a} + \underline{b}) = 7$

$\underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{b} = 7$

$|\underline{a}|^2 + \underline{a} \cdot \underline{b} = 7$

$5^2 + \underline{a} \cdot \underline{b} = 7$

$\underline{a} \cdot \underline{b} = 7 - 25$

$\underline{a} \cdot \underline{b} = -18$ D

18. $f(x) < 0$ for $s < x < t$

TRUE

$f'(x) < 0$ for $x < q$

FALSE

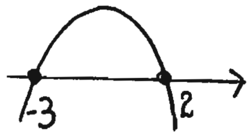
Gradient at $x = 0$ is 0 due to the stationary point.

B

Note: stat point (point of inflexion)

19. Graph $y = 6 - x - x^2$

$y = (3+x)(2-x)$



Roots $(3+x)(2-x) = 0$

$x = -3$ or $x = 2$

Solve $6 - x - x^2 < 0$ (B)

From graph: $x < -3$ and $x > 2$

20. $\dfrac{\log_b 9a^2}{\log_b 3a}$

$= \dfrac{\log_b (3a)^2}{\log_b 3a}$

$= \dfrac{2\cancel{\log_b 3a}}{\cancel{\log_b 3a}}$

$= 2$ (A)

Part B

21. i)

$$\begin{array}{c|cccc} 4 & 1 & -5 & 2 & 8 \\ \hline & \downarrow & 4 & -4 & -8 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

Since $f(4) = 0$
$(x-4)$ is a factor.

ii) $x^3 - 5x^2 + 2x + 8 = (x-4)(x^2 - x - 2)$

$= (x-4)(x-2)(x+1)$

iii) $x^3 - 5x^2 + 2x + 8 = 0$

$(x-4)(x-2)(x+1) = 0$

$\underline{x = -1, 2, 4}$

(b) $\displaystyle\int_0^2 (x^3 - 5x^2 + 2x + 8)\, dx$

$= \left[\dfrac{x^4}{4} - \dfrac{5x^3}{3} + x^2 + 8x\right]_0^2$

$= \left(\dfrac{(2)^4}{4} - \dfrac{5(2)^3}{3} + 2^2 + 8(2)\right) - (0)$

$= \dfrac{16}{4} - \dfrac{40}{3} + 4 + 16$

$= 24 - \dfrac{40}{3}$

$= \dfrac{72}{3} - \dfrac{40}{3}$

$= \underline{\dfrac{32}{3} \text{ or } 10\tfrac{2}{3} \text{ unit}^2}$

22. (a) $\cos x - \sqrt{3}\sin x = k\cos(x+a)$

$= k\cos x\cos a - k\sin x\sin a$

Comparing coefficients

$k\cos a = 1 \longrightarrow k^2\cos^2 a = 1$

$[\,-k\sin a = -\sqrt{3}\,]$

$k\sin a = \sqrt{3} \longrightarrow k^2\sin^2 a = 3$

$k^2 = 4$

$k = 2$

✓S | A✓✓
T | C✓

$\tan a = \dfrac{k\sin a}{k\cos a} = \dfrac{\sqrt{3}}{1}$

$\tan a = \sqrt{3}$

$a = \dfrac{\pi}{3}$

(b) $y = \cos x - \sqrt{3}\sin x$

$y = 2\cos(x + \frac{\pi}{3})$

Cuts y-axis : $x = 0$

$y = 2\cos(0 + \frac{\pi}{3})$

$y = 2 \times \frac{1}{2} = 1$

$(0, 1)$

Cuts x-axis : $y = 0$

$2\cos(x + \frac{\pi}{3}) = 0$

$\cos(x + \frac{\pi}{3}) = 0$

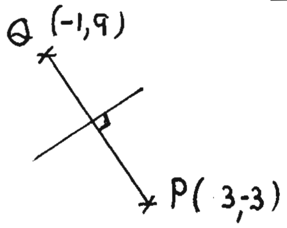
$x + \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{\pi}{2} - \frac{\pi}{3}, \frac{3\pi}{2} - \frac{\pi}{3}$

$x = \frac{\pi}{6}, \frac{7\pi}{6}$

$(\frac{\pi}{6}, 0)$ $(\frac{7\pi}{6}, 0)$

23.



$m_{PQ} = \dfrac{-3-9}{3-(-1)}$

$= \dfrac{-12}{4}$

$m_{PQ} = -3$

$m_{\text{perp. bisector}} = \frac{1}{3}$ (since $-3 \times \frac{1}{3} = -1$)

Midpoint of PQ $= \left(\frac{-1+3}{2}, \frac{9+(-3)}{2}\right)$

$M = (1, 3)$

Equation of L_1

$m = \frac{1}{3}$ passes thro' $M(1, 3) = (a, b)$

$y - b = m(x - a)$

$y - 3 = \frac{1}{3}(x - 1)$

$3y - 9 = x - 1$

$x - 3y = -8$ ①

23(b) Equation of L_2

$m_{\parallel \ell\ell} = m_{PQ} = -3$ and passes thro' $R(1,-2)$

$$y - b = m(x - a)$$
$$y + 2 = -3(x - 1)$$
$$y + 2 = -3x + 3$$
$$3x + y = 1 \quad (2)$$

(c) Solve
$$x - 3y = -8 \quad (1)$$
$$3x + y = 1 \quad (2)$$

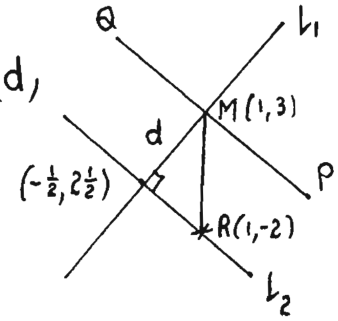
$3 \times (2)$
$$9x + 3y = 3 \quad (3)$$

$(1) + (3)$
$$10x = -5$$
$$x = -\tfrac{1}{2}$$

When $x = -\frac{1}{2}$
$$3 \times (-\tfrac{1}{2}) + y = 1$$
$$-1\tfrac{1}{2} + y = 1$$
$$y = 2\tfrac{1}{2}$$

Point of intersection $(-\frac{1}{2}, 2\frac{1}{2})$

(d)



Shortest distance

$$d = \sqrt{(1 - (-\tfrac{1}{2}))^2 + (3 - 2\tfrac{1}{2})^2}$$
$$d = \sqrt{(\tfrac{3}{2})^2 + (\tfrac{1}{2})^2}$$
$$d = \sqrt{\tfrac{10}{4}}$$
$$d = \frac{\sqrt{10}}{2} \text{ units.}$$

Mathematics Higher Paper 2

1 (a) i,
$$f(g(x)) = f(x+4) = (x+4)^2 + 3 = x^2+8x+19$$

ii,
$$g(f(x)) = g(x^2+3) = x^2+3+4 = x^2+7$$

(b,
$$f(g(x)) + g(f(x)) = 0$$
$$x^2+8x+19+x^2+7 = 0$$
$$2x^2+8x+26 = 0$$
$$x^2+4x+13 = 0$$

$a = 1 \quad b = 4 \quad c = 13$

disc $b^2-4ac = 4^2 - 4\times1\times13$
$= 16-52$
$= -36$

$b^2-4ac < 0$

$\Rightarrow$ No real roots.

2 (a) Line $2x-y+5=0 \Rightarrow y = 2x+5$

Circle $x^2+y^2-6x-2y-30=0$

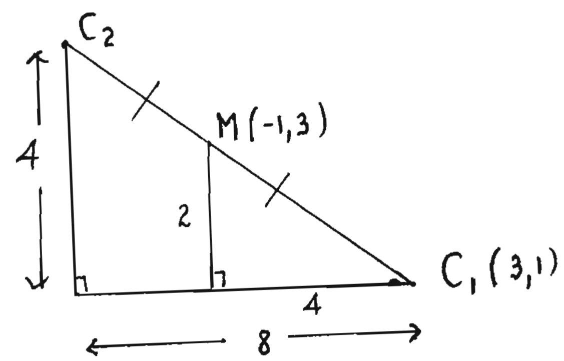
At the point of intersection
$$x^2+(2x+5)^2-6x-2(2x+5)-30 = 0$$
$$x^2+4x^2+20x+25-6x-4x-10-30 = 0$$
$$5x^2+10x-15 = 0$$
$$5(x^2+2x-3) = 0$$
$$5(x+3)(x-1) = 0$$
$$x+3=0 \text{ or } x-1=0$$
$$x = -3 \text{ or } 1$$

When $x = -3$: $y = 2\times(-3)+5$, $y = -1$, $(-3,-1)$

$x = 1$: $y = 2\times1+5$, $y = 7$, $(1,7)$

$P(-3,-1) \quad Q(1,7)$

2 (b)



Circle 1.

$$x^2+y^2-6x-2y-30=0$$
$$x^2+y^2+2gx+2fy+c=0$$

$2g=-6$ $\quad$ $2f=-2$ $\quad$ $c=-30$

$g=-3$ $\quad$ $f=-1$

Centre $C_1(-g,-f) \rightarrow C_1(3,1)$

Radius $r_1=\sqrt{g^2+f^2-c}$

$r_1=\sqrt{(-3)^2+(-1)^2-(-30)}$

$r_1=\sqrt{9+1+30}$

$r_1=\sqrt{40}$

$r_1=2\sqrt{10}$

Midpoint of PQ

M has coordinates $\left(\frac{-3+1}{2}, \frac{-1+7}{2}\right)$

$M(-1,3)$

Using method of stepping out

Circle 2.

$x = 3-8 = -5$

$y = 1+4 = 5$

Centre $C_2(-5,5)$

Equation of Circle 2.

$$(x-a)^2+(y-b)^2=r^2$$

Where $C_2(-5,5)=(a,b)$ $\quad$ $r_2=\sqrt{40}$

$$(x+5)^2+(y-5)^2=40$$

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3. Max. or Min. Values of f occur either at the Stationary Points or at the endpoints of the interval.

$$f(x)=x^3-2x^2-4x+6$$

Endpoints $\quad 0 \leq x \leq 3$

$f(0) = 0^3-2(0)^2-4(0)+6 = 6$

$f(3) = 3^3-2(3)^2-4(3)+6$

$= 27-18-12+6$

$= 3$

Stationary Points occur when $f'(x)=0$

$3x^2-4x-4=0$

$(3x+2)(x-2)=0$

$3x+2=0$ or $x-2=0$

$x \neq -\frac{2}{3}$ $\quad$ $x=2$

$0 \leq x \leq 3$

$f(2) = 2^3-2(2)^2-4(2)+6$

$= 8-8-8+6$

$= -2$

3.

Nature Table

x	$\overset{1}{\rightarrow}$	2	$\overset{2\frac{1}{2}}{\rightarrow}$
$f'(x)$	$-$	0	$+$
Shape	$\searrow$	$\rightarrow$	$\nearrow$

Min TP at $(2,-2)$

$x=1$

$f'(1) = (3+2)(1-2) = -ve$

$x = 2\frac{1}{2}$

$f'(2\frac{1}{2}) = (7\frac{1}{2}+2)(2\frac{1}{2}-2) = +ve$

Hence Max. value of f is 6 and
Min value of f is -2.

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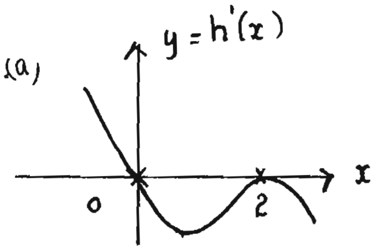
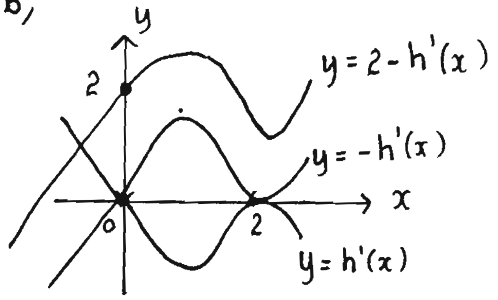
4 (a)



SPs. occurs when $x=0$ and 2

x	$\rightarrow$	0	$\rightarrow$	2	$\rightarrow$
$h'(x)$	$+$	0	$-$	0	$-$

(b)



$y = 2 - h'(x)$

$y = -h'(x) + 2$ ← translation 2 unit ↑ up

↑ Reflection in the X-axis

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5. (a) i)

$\overrightarrow{BA} = \underline{a} - \underline{b}$

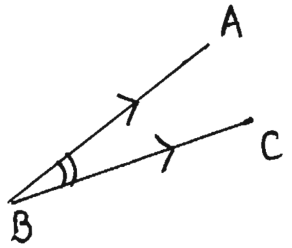
$= \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\overrightarrow{BC} = \underline{c} - \underline{b}$

$= \begin{pmatrix} 4 \\ k \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ k+3 \\ -1 \end{pmatrix}$



$$\cos A\hat{B}C = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$$

$$\cos A\hat{B}C = \frac{3}{\sqrt{2} \times \sqrt{k^2+6k+14}}$$

$$\cos A\hat{B}C = \frac{3}{\sqrt{2(k^2+6k+14)}}$$

as required.

$|\overrightarrow{BA}| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$

$|\overrightarrow{BC}| = \sqrt{2^2 + (k+3)^2 + (-1)^2}$

$= \sqrt{5 + k^2 + 6k + 9}$

$|\overrightarrow{BC}| = \sqrt{k^2 + 6k + 14}$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = 1 \times 2 + 0 \times (k+3) + (-1) \times (-1)$

$= 2 + 1$

$= 3$

5(b) $A\hat{B}C = 30^\circ$

$\cos A\hat{B}C = \cos 30^\circ = \frac{\sqrt{3}}{2}$

Solve $\frac{\sqrt{3}}{2} = \frac{3}{\sqrt{2(k^2+6k+14)}}$

squaring both sides

$\frac{3}{4} = \frac{9}{2(k^2+6k+14)}$

cross multiplication

$6(k^2+6k+14) = 36$

$k^2+6k+14 = 6$

$k^2+6k+8 = 0$

$(k+4)(k+2) = 0$

$k = -4, -2$

6.(a) $U_{n+1} = (\sin x)U_n + \cos 2x$

where $U_0 = 1$

$[U_{n+1} = aU_n + b]$

For $0 < x < \frac{\pi}{2}$, $0 < \sin x < 1$

$\Rightarrow$ Sequence has a limit.

(b) Limit $= \frac{b}{1-a} = \frac{\cos 2x}{1-\sin x}$

Given Limit is $\frac{1}{2}\sin x$

then $\frac{\cos 2x}{1-\sin x} = \frac{1}{2}\sin x$

$\frac{1}{2}\sin x(1-\sin x) = \cos 2x$

$\frac{1}{2}\sin x - \frac{1}{2}\sin^2 x = 1 - 2\sin^2 x$

$\frac{3}{2}\sin^2 x + \frac{1}{2}\sin x - 1 = 0$

x 2

$3\sin^2 x + \sin x - 2 = 0$

$(3\sin x - 2)(\sin x + 1) = 0$

$3\sin x - 2 = 0$ or $\sin x + 1 = 0$

$\sin x = \frac{2}{3}$ or $\sin x = -1$

Radian Mode $\sin^{-1}(\frac{2}{3})$

$x = 0{\cdot}73$ radians.

$\cancel{x = \frac{3\pi}{2}}$

$0 < x < \frac{\pi}{2}$

7. (a) At the point of intersection

$$4^x = 3^{2-x}$$

$$4^x = \frac{3^2}{3^x}.$$

Taking $\log_a$ of both sides

$$\log_a 4^x = \log_a \frac{3^2}{3^x}$$

$$\log_a 4^x = \log_a 9 - \log_a 3^x$$

$$x \log_a 4 + x \log_a 3 = \log_a 9$$

$$x(\log_a 4 + \log_a 3) = \log_a 9$$

$$x = \frac{\log_a 9}{\log_a 12}$$

(b) Using Natural logs.

$$x = \frac{\ln 9}{\ln 12} = 0.8842$$

$$y = 4^x = 4^{0.8842} = 3.41 \text{ to 2dp.}$$